ON AMENABILITY OF GROUP ALGEBRAS, I

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Abstract. We study amenability of algebras and modules (based on the notion of almost-invariant finite-dimensional subspace), and apply it to algebras associated with finitely generated groups.

We show that a group G is amenable if and only if its group ring $\mathbb{K}G$ is amenable for some (and therefore for any) field $\mathbb{K}$.

Similarly, a G-set X is amenable if and only if its span $\mathbb{K}X$ is amenable as a $\mathbb{K}G$-module for some (and therefore for any) field $\mathbb{K}$.

1. Introduction

Amenability of groups was introduced in 1929 by Von Neumann [7]:

Definition 1.1. A (discrete) group G is *amenable* if it admits a measure $\mu : 2^G \to [0,1]$ such that $\mu(G) = 1$ and $\mu(A \sqcup B) = \mu(A) + \mu(B)$ and $\mu(Ag) = \mu(A)$ for all disjoint $A, B \subseteq G$ and $g \in G$.

This notion may serve as a witness to the "structure" of groups: either a group is *amenable*, in which case it admits a right-translation invariant finitely additive measure, or it is *non-amenable*, in which case it admits a "paradoxical" decomposition in finitely many pieces, which can be reassembled by left-translation in two copies of the original group; see [13]. More generally:

Definition 1.2. Let G be a group acting on the right on a set X. This action is *amenable* if there exists a measure $\mu : 2^X \to [0,1]$ such that $\mu(X) = 1$ and $\mu(A \sqcup B) = \mu(A) + \mu(B)$ and $\mu(Ag) = \mu(A)$ for all disjoint $A, B \subseteq X$ and $g \in G$.

Under this definition, a group G is amenable if its action on itself by right-multiplication is amenable. This definition will be reformulated in terms of Følner sets (see Lemma 3.1).

1.1. Amenable algebras.

The present note explores the notion of amenability for associative algebras, which appeared in [1, 4]. Throughout this note, $\mathbb{K}$ denotes an arbitrary field — although the results easily extend to integral domains. We shall actually phrase it in the more natural language of modules:

Definition 1.3. Let R be an associative algebra, and let M be a right R-module. It is *amenable* if, for every $\epsilon > 0$ and every finite-dimensional subspace S of R, there exists a finite-dimensional subspace F of M such that

$$\frac{\dim_{\mathbb{K}}((F + Fs)/F)}{\dim_{\mathbb{K}}(F)} < \epsilon \text{ for all } s \in S.$$

The same definition holds, *mutatis mutandis*, for left modules.

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The main result of this note is the following, proved in §3:

Theorem 1.4. *Let $\mathbb{K}$ be any field, and let X be a right G-set. Then X is amenable if and only if its linear span $\mathbb{K}X$ is amenable.*

Letting G act on itself by right-multiplication, we obtain:

Corollary 1.5. *Let $\mathbb{K}$ be any field, and let G be a group. Then G is amenable if and only if its group algebra $\mathbb{K}G$ is amenable.*

The "only if" part of the corollary is claimed in [1], where the "if" part is proven in case $\mathbb{K} = \mathbb{C}$. M. Gromov pointed out to me that the "if" part admits a simple proof if $\mathbb{K}$ has characteristic 0.

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2. Convex sets

We recall the notion of *Steiner point* of a convex polytope [6, §14.3]. Let P be a convex polytope in $\mathbb{R}^n$. For $x \in P$ set

$$C(x, P) = \{v \in \mathbb{S}^{n-1} : \langle x' - x | v\rangle \geq 0 \text{ for all } x' \in P\};$$

this is the set of outer normal vectors of half-spaces containing P and with x on their boundary. Let $\measuredangle(x, P)$ denote the normalized content of $C(x, P)$:

$$\measuredangle(x, P) = \frac{\lambda(C(x, P))}{\lambda(\mathbb{S}^{n-1})}, \quad \text{where } \lambda \text{ denotes Lebesgue measure.}$$

For obvious geometric reasons the number $\measuredangle(x, P)$ is called the *exterior angle* of P at x.

Recall that the *Minkowski sum* of two polytopes P, Q is the polytope $P \dot{+} Q = \{x + y : x \in P, y \in Q\}$.

Lemma 2.1. $C(x + y, P \dot{+} Q) = C(x, P) \cap C(y, Q)$.

Proof.

$$\begin{aligned} C(x + y, P \dot{+} Q) &= \{v \in \mathbb{S}^{n-1} : \langle x' + y' - (x + y) | v\rangle \geq 0 \text{ for all } x' \in P, y' \in Q\} \\ &= \{v \in \mathbb{S}^{n-1} : \langle x' - x | v\rangle \geq 0 \text{ for all } x' \in P \\ &\qquad\qquad \text{and } \langle y' - y | v\rangle \geq 0 \text{ for all } y' \in Q\} \\ &= C(x, P) \cap C(y, Q). \end{aligned}$$

□

Let V denote the set of extremal points of P; then $\measuredangle(x, P)$ is non-zero if and only if $x \in V$. The *Steiner point* of P is

$$m(P) = \sum_{x \in V} \measuredangle(x, P) x. \tag{1}$$

Up to measure-zero sets, $\{C(x, P) : x \in V\}$ is a partition of $\mathbb{S}^{n-1}$, so $\sum_{x \in P} \measuredangle(x, P) = 1$ and thus $m(V) \in V$.

Proposition 2.2 ([10])**.** *The function m is the only continuous $\mathbb{R}^n$-valued function on convex polytopes in $\mathbb{R}^n$ that satisfies $m(\alpha A \dot{+} (1-\alpha)B) = \alpha m(A) + (1-\alpha)m(B)$ for any convex polytopes A, B and $\alpha \in [0,1]$ and $m(gA) = gm(A)$ for any similarity $g : \mathbb{R}^n \to \mathbb{R}^n$.*

Let F be a subspace of the vector space $\mathbb{K}^n$. For any $S \subseteq \{1, \dots, n\}$, let $\pi_S : \mathbb{K}^n \to \mathbb{K}^S$ denote the projection $(v_1, \dots, v_n) \mapsto (v_i)_{i \in S}$. Define

$$X_F = \big\{ S : \pi_S \text{ restricts to an isomorphism } F \to \mathbb{K}^S \big\}. \tag{2}$$

Let e_i be the ith basis vector in $\mathbb{R}^n$, and set

$$V_F = \Big\{ \sum_{i \in S} e_i : S \in X_F \Big\}, \quad P_F = \text{the convex hull of } V_F, \quad m_F = m(P_F). \tag{3}$$

Lemma 2.3. *All the $v \in V_F$ are $\{0,1\}$-vectors. The sets X_F and V_F are non-empty, and P_F is a non-empty, closed, convex polytope in $[0,1]^n$.*

Proof. The only non-trivial statements are that X_F, and therefore V_F and P_F, are non-empty. Let S be maximal such that π_S restricts to a surjection $F \to \mathbb{K}^S$. If $\pi_S|F$ were not injective, there would be $v \neq 0$ in $\ker(\pi_S|F)$; let $k \in \{1, \dots, n\}$ be a non-zero coëfficient of v; then $k \notin S$ and $\pi_{S \cup \{k\}}$ is surjective from F onto $\mathbb{K}^{S \cup \{k\}}$, since its image contains $0 \times \mathbb{K}^{\{k\}}$ and projects onto $\mathbb{K}^S$. This contradicts the maximality of S. We therefore have $S \in X_F$. □

The proof of Theorem 1.4 hinges on the following

Proposition 2.4. *Let $E \leq F \leq \mathbb{K}^n$ be subspaces. Then $m_E \leq m_F$ coördinate-wise.*

Lemma 2.5. *Let $E \leq F \leq \mathbb{K}^n$ be subspaces. Then*

1. *for every $S \in X_E$ there exists $T \in X_F$ with $S \subseteq T$;*
2. *for every $T \in X_F$ there exists $S \in X_E$ with $S \subseteq T$;*
3. *for every $S \in X_E, T \in X_F$ and $k \in S$ there exists $\ell \in T$ with $S \setminus \{k\} \cup \{\ell\} \in X_E$ and $T \setminus \{\ell\} \cup \{k\} \in X_F$.*

Proof. (1) Consider $D = \ker(\pi_S) \cap F$. By Lemma 2.3, there exists $U \subset \{1, \dots, n\}$ such that $\pi_U : D \to \mathbb{K}^U$ is an isomorphism. Clearly $U \cap S = \emptyset$, so $T = S \sqcup U \in X_F$.

(2) Apply Lemma 2.3 to the inclusion $\pi_T(E) \leq \mathbb{K}^T$.

(3) Let $(e_i)_{i \in \{1,\dots,n\}}$ be the standard basis of $\mathbb{K}^n$. Choose a basis $(\epsilon_i)_{i \in S}$ of E such that $\langle \epsilon_i | e_j \rangle = \delta_{ij}$ for all $i, j \in S$, and choose a basis $(\phi_i)_{i \in S}$ of F such that $\langle \phi_i | e_j \rangle = \delta_{ij}$ for all $i, j \in T$.

Since $E \leq F$, we may write $\epsilon_k = \sum_{\ell \in T} \alpha_\ell \phi_\ell$; and for all $\ell \in T$ we have $\langle \epsilon_k | e_\ell \rangle = \sum_{\ell' \in T} \alpha_{\ell'} \langle \phi_{\ell'} | e_\ell \rangle = \alpha_\ell$. Therefore

$$1 = \langle \epsilon_k | e_k \rangle = \sum_{\ell \in T} \alpha_\ell \langle \phi_\ell | e_k \rangle = \sum_{\ell \in T} \langle \epsilon_k | e_\ell \rangle \langle \phi_\ell | e_k \rangle;$$

so $\langle \epsilon_k | e_\ell \rangle \langle \phi_\ell | e_k \rangle \neq 0$ for some $\ell \in T$. This implies that $\langle \epsilon_k | e_\ell \rangle \neq 0$, so $\pi_{S \setminus \{k\} \cup \{\ell\}} : E \to \mathbb{K}^{S \setminus \{k\} \cup \{\ell\}}$ is an isomorphism: its image surjects onto $\mathbb{K}^{S \setminus \{k\}}$, and contains $0 \times \mathbb{K}^{\{\ell\}} = \pi_{S \setminus \{k\} \cup \{\ell\}}(\mathbb{K}\epsilon_k)$. Since $\pi_{S \setminus \{k\} \cup \{\ell\}}$ maps onto a space of dimension $\#S$, it is an isomorphism. We also have $\langle \phi_\ell | e_k \rangle \neq 0$, which by the same argument implies that $\pi_{T \setminus \{\ell\} \cup \{k\}} : F \to \mathbb{K}^{T \setminus \{\ell\} \cup \{k\}}$ is an isomorphism. □

Proof of Proposition 2.4. For $\varepsilon \in [0,1]$, let $P_\varepsilon = (1-\varepsilon)P_E \dot{+} \varepsilon P_F$ be the Minkowski linear combination of P_E and P_F. It is the convex envelope of $(1-\varepsilon)V_E + \varepsilon V_F$. Set

$$V_\varepsilon = \{(1-\varepsilon)x + \varepsilon y : x \in V_E, y \in V_F, \text{ and } x \le y \text{ coördinatewise}\}.$$

Lemma 2.6. *P_ε is the convex envelope of V_ε.*

Proof. If $\varepsilon \in \{0,1\}$ this follows from Lemma 2.5(1),(2). Consider then $\varepsilon \in (0,1)$ and $x \in V_E, y \in V_F$ with $x \not\le y$. By Lemma 2.5(3) there exist k, ℓ such that $x' := x - e_k + e_\ell \in V_E$ and $y' := y - e_\ell + e_k \in V_F$. Furthermore $k \ne \ell$ because $x \not\le y$. Now

$$(1-\varepsilon)x + \varepsilon y = (1-\varepsilon)\big((1-\varepsilon)x + \varepsilon x'\big) + \varepsilon\big(\varepsilon y + (1-\varepsilon)y'\big)$$

is a convex combination of non-extremal points of P_E and P_F, so is not an extremal point of P_ε. □

Suppose now $\varepsilon \in (0,1)$. Let $\alpha_\varepsilon : V_\varepsilon \to V_E$ be the map $(1-\varepsilon)x + \varepsilon y \mapsto x$; it truncates non-1 coördinates down to 0, so $\alpha(z) \le z$ coördinatewise for all $z \in V_\varepsilon$. By Lemma 2.5(1) this map is onto. By Lemma 2.6 we have $\lambda(C((1-\varepsilon)x + \varepsilon y, P_\varepsilon) = 0$ if $y \not\ge x$. By Lemma 2.1 we compute

$$\begin{aligned}\sum_{z \in \alpha_\varepsilon^{-1}(x)} \measuredangle(z, P_\varepsilon) &= \sum_{z \in \alpha_\varepsilon^{-1}(x)} \frac{\lambda(C(z, P_\varepsilon))}{\lambda(\mathbb{S}^{n-1})} = \sum_{y \in V_F} \frac{\lambda(C(x, P_E) \cap C(y, P_F))}{\lambda(\mathbb{S}^{n-1})} \\ &= \frac{\lambda(C(x, P_E))}{\lambda(\mathbb{S}^{n-1})} = \measuredangle(x, P_E).\end{aligned}$$

We conclude

$$\begin{aligned} m(P_\varepsilon) &= \sum_{z \in V_\varepsilon} \measuredangle(z, P_\varepsilon) z = \sum_{x \in V_E} \sum_{z \in \alpha_\varepsilon^{-1}(x)} \measuredangle(z, P_\varepsilon)(x + (z - x)) \\ &= \sum_{x \in V_E} \measuredangle(x, P_E)x + \sum_{z \in V_\varepsilon} \measuredangle(z, P_\varepsilon)(z - x) \\ &= m(P_E) + \text{something non-negative} \ge m_E \text{ coördinatewise.}\end{aligned}$$

The conclusion holds for $m(P_1) = m_F$ by continuity of m, see Proposition 2.2. □

Note that $\#S = \dim_{\mathbb{K}} F$ for all $S \in X_F$, and $\|x\|_1 = \dim_{\mathbb{K}} F$ for all $x \in V_F$, so $\|m_F\|_1 = \dim_{\mathbb{K}} F$.

Corollary 2.7. *Let $E \le F \le \mathbb{K}^n$ be subspaces. Then $\|m_F - m_E\|_1 = \dim_{\mathbb{K}}(F/E)$.*

Proof. By Proposition 2.4,

$$\|m_F - m_E\|_1 = \|m_F\|_1 - \|m_E\|_1 = \dim_{\mathbb{K}}(F) - \dim_{\mathbb{K}}(E) = \dim_{\mathbb{K}}(F/E). \qquad \square$$

3. Proof of Theorem 1.4

We recall that there are sundry equivalent definitions of amenability for G-sets:

Lemma 3.1. *Let G be a group and let X be a right G-set. The following are equivalent:*

(1) *X is amenable;*

(2) *for every $\epsilon > 0$ and every finite subset S of G, there exists a finite subset F of X such that*
$$\frac{\#(F \cup FS) - \#F}{\#F} < \epsilon;$$
(3) *for every $\epsilon > 0$ and every finite subset S of G, there exists a finite subset F of X such that*
$$\frac{\#(F \cup Fs) - \#F}{\#F} < \epsilon \text{ for all } s \in S;$$
(4) *for every $\epsilon > 0$ and every finite subset S of G, there exists $f : X \to \mathbb{R}_+$, with finite support, such that*
$$\frac{\|f - fs\|_1}{\|f\|_1} < \epsilon \text{ for all } s \in S.$$

The equivalence between (2), (3) and (4) is classical, see e.g. [8, Theorems 4.4, 4.10, 4.13]. The equivalence of these with (1) is proven there in the case $X = G$; see also [9].

Similarly, there are various equivalent definitions of amenability for modules:

Lemma 3.2. *Let R be an affine algebra and let M be a right module. The following are equivalent:*

(1) *M is amenable;*
(2) *for every $\epsilon > 0$ and every finite-dimensional subspace S of R, there exists a finite-dimensional subspace F of M, such that*
$$\frac{\dim_{\mathbb{K}}((F + FS)/F)}{\dim_{\mathbb{K}}(F)} < \epsilon.$$

Proof. Assume first that M is amenable. Let there be given $\epsilon > 0$ and a finite-dimensional subspace $S \leq R$. Let F be a finite-dimensional subspace of M such that $\dim_{\mathbb{K}}(F + Fs) < (1 + \epsilon/\dim_{\mathbb{K}} S) \dim_{\mathbb{K}} F$ for all $s \in S$. Then $\dim_{\mathbb{K}}(F + FS) < (1 + \epsilon) \dim_{\mathbb{K}} F$, so (2) holds. The converse implication is trivial. □

Proof of Theorem 1.4. Suppose first that X is amenable. Let $\epsilon > 0$ be given, and let S be a finite-dimensional subspace of $\mathbb{K}G$. Let S' be the support of S, i.e. the union of the supports of all elements of S; it is a finite subset of G. By Lemma 3.1(2) there exists a finite subset F' of X with $(\#(F' \cup F'S') - \#F')/\#F' < \epsilon$. Set $F = \mathbb{K}F'$, a finite-dimensional subspace of $\mathbb{K}X$. We have $\dim_{\mathbb{K}} F = \#F'$ and $\dim_{\mathbb{K}}(FS) \leq \#F'S'$, so $\dim_{\mathbb{K}}(F + FS) \leq \#(F' \cup F'S')$, whence
$$\frac{\dim_{\mathbb{K}}((F + FS)/F)}{\dim_{\mathbb{K}}(F)} < \epsilon,$$
so $\mathbb{K}X$ is amenable by Lemma 3.2(2).

Suppose now that the $\mathbb{K}G$-module $\mathbb{K}X$ is amenable. Let $\epsilon > 0$ be given, and let S be a finite subset of G. Set $S' = \mathbb{K}S$ and, using Lemma 3.2(1), let F be a finite-dimensional subspace of $\mathbb{K}X$ such that $\dim_{\mathbb{K}}((F + Fs)/F)/\dim_{\mathbb{K}}(F) < \frac{\epsilon}{2}$ for all $s \in S$. Set $f = m_F$ as defined in (3), page 3. We have $\dim_{\mathbb{K}}((F + Fs)/F) < \frac{\epsilon}{2} \dim_{\mathbb{K}}(F)$, so $\|m_{F+Fs} - m_F\|_1 < \frac{\epsilon}{2} \dim_{\mathbb{K}}(F)$ by Corollary 2.7; and similarly $\|m_{F+Fs} - m_{Fs}\|_1 <$

$\frac{\epsilon}{2}\dim_{\mathbb{K}}(Fs)$. Now $\dim_{\mathbb{K}} F = \dim_{\mathbb{K}}(Fs) = \|f\|_1$, so we get

$$\begin{aligned}\|f - fs\|_1 = \|m_F - m_{Fs}\|_1 &\le \|m_{F+Fs} - m_F\|_1 + \|m_{F+Fs} - m_{Fs}\|_1\\ &< \frac{\epsilon}{2}\|f\|_1 + \frac{\epsilon}{2}\|f\|_1 = \epsilon\|f\|_1,\end{aligned}$$

and therefore X is amenable by Lemma 3.1(4). □

4. Exhaustively amenable sets and modules

The original definition of amenability for algebras was formulated slightly differently [4, §1.11]. We show here that it is equivalent to Definition 1.3 for group algebras.

Definition 4.1. A right G-set X is *exhaustively amenable* if there exists an increasing net $(F_\lambda)_{\lambda\in\Lambda}$ of finite subsets of X such that $\bigcup_{\lambda\in\Lambda} F_\lambda = X$ and for all $g \in G$:

$$\lim_{\lambda\in\Lambda} \frac{\#(F_\lambda \cup F_\lambda g)}{\# F_\lambda} = 1.$$

Lemma 4.2. *Let G be a group and let X be a right G-set.*

(1) *X is exhaustively amenable if and only if for every $\epsilon > 0$ and all finite sets $S \subseteq G$ and $U \subseteq X$ there exists a finite subset $F \subseteq X$ such that*

$$\frac{\#(F \cup FS)}{\#F} < 1 + \epsilon.$$

(2) *If X is exhaustively amenable, then it is amenable.*

(3) *If X is amenable and has no finite orbit, then it is exhaustively amenable.*

Proof. (1) Assume that X is exhaustively amenable, exhausted by a net $(F_\lambda)_{\lambda\in\Lambda}$. Let there be given $\epsilon > 0$ and finite subsets $S \subseteq G$, $U \subseteq M$. Let λ be large enough so that F_λ contains U and $\#(F_\lambda \cup F_\lambda s) < (1 + \frac{\epsilon}{\#S})\#F_\lambda$ for all $s \in S$. Then $\#(F_\lambda \cup F_\lambda S) < (1 + \epsilon)\#F_\lambda$.

Assume then the converse. Let $(g_\lambda)_{\lambda\in\Lambda'}$ be a well-ordering of G. Let $(x_\lambda)_{\lambda\in\Lambda''}$ be a well-ordering of X. Set $\Lambda = \Lambda' \times \Lambda''$ with the product order; set $g_{(\lambda',\lambda'')} = g_{\lambda'}$ and $x_{(\lambda',\lambda'')} = x_{\lambda''}$. Let $\epsilon : \Lambda \to \mathbb{R}_+$ be a decreasing function with $\lim_{\lambda\in\Lambda} \epsilon_\lambda = 0$. For every $\lambda \in \Lambda$, let F_λ be a finite subset of X, containing F_μ and x_μ for all $\mu < \lambda$, and such that $\#(F_\lambda \cup F_\lambda g_\mu) < (1+\epsilon_\lambda)\#F_\lambda$ for all $\mu < \lambda$. This is an exhausting sequence of asymptotically invariant subspaces, showing that X is exhaustively amenable.

(2) follows clearly from (1).

(3) Let $(g_\lambda)_{\lambda\in\Lambda}$ be a well-ordering of G. Let $\epsilon : \Lambda \to \mathbb{R}_+$ be a decreasing function with $\lim_{\lambda\in\Lambda} \epsilon_\lambda = 0$. For every $\lambda \in \Lambda$, let F_λ be a finite subset of X, such that $\#(F_\lambda \cup F_\lambda g_\mu) < \epsilon_\lambda \#F_\lambda$ for all $\mu < \lambda$.

If $\#F_\lambda$ is unbounded, let $S \subseteq G$ and $U \subseteq X$ be finite subsets, and let $\epsilon > 0$ be given. Let λ be large enough so that $\epsilon_\lambda \le \frac{\epsilon}{2\#S}$ and $\max\{\mu : g_\mu \in S\} < \lambda$ and $\#F_\lambda \ge \frac{2}{\epsilon}\#(U \cup US)$. Set $F = F_\lambda \cup U$; then

$$\frac{\#(F \cup FS)}{\#F} \le \frac{\#(F_\lambda \cup \mathbb{F}_{p\lambda} S) + \#(U \cup US)}{\#F_\lambda} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so X is exhaustively amenable by (1).

Assume therefore that $\#F_\lambda \le m$ for all $\lambda \in \Lambda$. Then $F_\lambda g_\mu = F_\lambda$ for all $\lambda > \mu$, as soon as $\epsilon_\mu < \frac{1}{m}$. Set $F_\infty = \bigcup_{\lambda\in\Lambda} F_\lambda$.

If F_∞ is infinite, let $N : \Lambda \to \mathbb{N}$ be an increasing function with $\lim_{\lambda\in\Lambda} N_\lambda = \infty$. For all $\lambda \in \Lambda$, the set $\bigcup_{\mu>\lambda} F_\mu$ is infinite. Let $\widetilde{F}_\lambda$ be a union of finitely many F_μ with $\mu \geq \lambda$, such that $\#\widetilde{F}_\lambda \geq N_\lambda$. Then we still have $\widetilde{F}_\lambda g_\mu = \widetilde{F}_\lambda$ for all $\lambda > \mu$, as soon as $\epsilon_\mu < \frac{1}{m}$. We are back in the case "$\#F_\lambda$ unbounded".

Finally, if F_∞ is finite, then there exists $F \subseteq F_\infty$ such that for every $\lambda \in \Lambda$, there exists $\mu \geq \lambda$ with $F_\mu = F$. This F is a finite G-orbit. □

The following definition generalizes [4, §1.11] to uncountable-dimensional algebras and to modules:

Definition 4.3. Let R be an algebra, and let M be a right R-module. It is *exhaustively amenable* if there exists an increasing net $(F_\lambda)_{\lambda\in\Lambda}$ of finite-dimensional subspaces of M such that $\bigcup_{\lambda\in\Lambda} F_\lambda = M$ and for all $r \in R$:

$$\lim_{\lambda\in\Lambda} \frac{\dim_{\mathbb{K}}(F_\lambda + F_\lambda r)}{\dim_{\mathbb{K}} F_\lambda} = 1.$$

Lemma 4.4. (1) *M is exhaustively amenable if and only if for every $\epsilon > 0$ and all finite-dimensional subspaces $S \leq R$ and $U \leq M$ there exists a finite-dimensional subspace $F \leq M$ such that*

$$\frac{\dim_{\mathbb{K}}(F + FS)}{\dim_{\mathbb{K}} F} < 1 + \epsilon.$$

(2) *If M is exhaustively amenable, then it is amenable.*

(3) *If the $\mathbb{K}G$-module $\mathbb{K}X$ is amenable and X has no finite orbits, then $\mathbb{K}X$ is exhaustively amenable.*

Proof. (1) Assume that M is exhaustively amenable, exhausted by a net $(F_\lambda)_{\lambda\in\Lambda}$. Let there be given $\epsilon > 0$ and finite-dimensional subspaces $S \leq R$, $U \leq M$. Choose a basis $(b_1, \dots, b_d)$ of S, and let λ be large enough so that F_λ contains U and $\dim_{\mathbb{K}}(F_\lambda + F_\lambda b_i) < (1 + \frac{\epsilon}{d}) \dim_{\mathbb{K}} F_\lambda$ for all $i \in \{1, \dots, d\}$. Then $\dim_{\mathbb{K}}(F_\lambda + F_\lambda S) < (1 + \epsilon) \dim_{\mathbb{K}} F_\lambda$.

Assume then the converse. Let $(r_\lambda)_{\lambda\in\Lambda'}$ be a well-ordered basis of R. Let $(m_\lambda)_{\lambda\in\Lambda''}$ be a well-ordered basis of M. Set $\Lambda = \Lambda' \times \Lambda''$ with the product order; set $r_{(\lambda',\lambda'')} = r_{\lambda'}$ and $m_{(\lambda',\lambda'')} = m_{\lambda''}$. Let $\epsilon : \Lambda \to \mathbb{R}_+$ be a decreasing function with $\lim_{\lambda\in\Lambda} \epsilon_\lambda = 0$. For every $\lambda \in \Lambda$, let F_λ be a finite-dimensional subspace of M, containing F_μ and m_μ for all $\mu < \lambda$, and such that $\dim_{\mathbb{K}}(F_\lambda + F_\lambda r_\mu) < (1 + \epsilon_\lambda) \dim_{\mathbb{K}}(F_\lambda)$ for all $\mu < \lambda$. This is an exhausting sequence of asymptotically invariant subspaces, showing that M is exhaustively amenable.

(2) follows clearly from (1).

(3) If $\mathbb{K}X$ is amenable, then X is amenable by Theorem 1.4; since it has no finite orbits, it is exhaustively amenable by Lemma 4.2(3). The first part of the proof of Theorem 1.4 extends easily to show that $\mathbb{K}X$ is exhaustively amenable. □

Corollary 4.5. *The following are equivalent:*

(1) *$\mathbb{K}G$ is amenable;*
(2) *$\mathbb{K}G$ is exhaustively amenable;*
(3) *G is amenable.*

Proof. (1) and (3) are equivalent by Theorem 1.4, and (1) follows from (2) by Lemma 4.4(2). If G is finite then there is nothing to prove; otherwise G, as a right G-set, has a single orbit, which is infinite, so Lemma 4.4(3) applies. □

5. Isoperimetric profile

There is a quantitative estimate of amenability, called the *isoperimetric profile* (see [11, §VI.1], [5, §5.E] and [12, page 325] for its first appearances): for G-sets X, this is the function

$$I_X(v,S)=\min_{\substack{F\subseteq X\\ \#F\le v}}\frac{\#(F\cup FS)-\#F}{\#F}.$$

Then by Lemma 3.1(2) amenability of G is equivalent to $\lim_{v\to\infty} I_G(v,S)=0$ for all finite $S\subseteq G$. Note that, following the equivalence between (2) and (4) in Lemma 3.1, we have

$$I_X(v,S)\sim \inf_{\substack{f\in\ell^1(X)\\ \#\operatorname{support}(f)\le v}}\max_{s\in S}\frac{\|f-fs\|_1}{\|f\|_1}, \tag{4}$$

where '$I(n,S)\sim J(n,S)$' means that, for any $S\subseteq G$, the quotient $I(n,S)/J(n,S)$ is bounded over all $n\in\mathbb{N}$.

If X is amenable, a better normalization of its isoperimetric profile (see [3] and [2]) is

$$\Phi_X(n,S)=\min\{v\in\mathbb{N}:\ I_X(v,S)\le 1/n\}.$$

For two functions $\Phi,\Psi:\mathbb{N}\to\mathbb{N}$ we write '$\Phi(n)\sim\Psi(n)$' to mean that there exists $K\in\mathbb{N}$ with $\Phi(n)\le\Psi(Kn)$ and $\Psi(n)\le\Phi(Kn)$ for all $n\in\mathbb{N}$. If $X=G$ is a finitely-generated group, then the equivalence class of $\Phi_G(n,S)$ is independent of the choice of generating set S of G, and is denoted $\Phi_G(n)$. For example, $\Phi_{\mathbb{Z}}(n)\sim n$.

The function $\Phi_X(n,S)$ is well-defined if and only if X is amenable. A general result is that Φ_X is at least as large as the growth function of X, see [11, §VI.1].

Similarly, for a right R-module M we define

$$I_M(v,S)=\min_{\substack{F\subseteq M\\ \dim_{\mathbb{K}}(F)\le v}}\frac{\dim_{\mathbb{K}}((F+FS)/F)}{\dim_{\mathbb{K}}F}. \tag{5}$$

Then by Lemma 3.2(2) amenability of M is equivalent to $\lim_{v\to\infty} I_M(v,S)=0$ for all finite-dimensional $S<R$. We also set

$$\Phi_M(n,S)=\min\{v\in\mathbb{N}:\ I_M(v,S)\le 1/n\}.$$

We then remark that the proof of Theorem 1.4 shows that $I_{\mathbb{K}X}(n,\mathbb{K}S)\le I_X(n,S)$ and $\Phi_{\mathbb{K}X}(n,\mathbb{K}S)\le\Phi_X(n,S)$.

On the other hand, let $G=(\mathbb{Z}/2\mathbb{Z})\wr\mathbb{Z}$ be the 'lamplighter group', generated for definiteness by $\pm1\in\mathbb{Z}$ and $\delta_0:\mathbb{Z}\to\mathbb{Z}/2$ the Dirac mass at 0. Then $\Phi_G(n)\sim 2^n n$: examples of subsets $F\subseteq G$ that achieve the minimum in (4) are of the form

$$F=\big\{(f,t)\in G:\ 1\le t\le n\text{ and }\operatorname{support}(f)\subseteq\{1,\dots,n\}\big\},$$

with $v=2^n n$ elements and $\#(F\cup FS)=\frac{n+2}{n}v$. Nevertheless, $\Phi_{\mathbb{K}G}(n)\sim n$: examples of subspaces $F\le\mathbb{K}G$ that achieve the minimum in (5) are of the form

$$F=\Big\langle\sum_{\substack{f:\mathbb{Z}\to\mathbb{Z}/2\mathbb{Z}\\ \operatorname{support}(f)\subseteq\{1,\dots,n\}}}(f,t):\ t\in\{1,\dots,n\}\Big\rangle,$$

of dimension $v=n$ and with $\dim_{\mathbb{K}}(F+FS)=n+2$.

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